Test 2 / Numerical Mathematics 1 / May 25th 2020, University of Groningen

A simple calculator is allowed.
No additional material is allowed.
All answers need to be justified using mathematical arguments.
Total time: 1 hour 30 minutes (time includes upload of the PDF with your answers to Nestor) + 10 minutes (if special needs)
Remember: oral "checks" may be run afterwards.
Grade = (obtained points) + 1

Exercise 1 (6 points)

Consider the function $f(x) = x^2 - x - 1$.

(a) $\boxed{1.0}$ Show that if x^* satisfies $f(x^*) = 0$, then for $g(x) = 1 + 1/x$, x^* satisfies $g(x^*) = x^*$. Propose, without using any derivative of $f(x)$, another function $h(x) \neq g(x)$ and $h(x) \neq 1/(x-1)$ verifying $h(x^*) = x^*$.

(b) $\boxed{1.0}$ Compute 3 fixed point iterations using $g(x)$ starting from $x^{(0)} = -N - 2$, with N the last digit of your student number.

(c) $\boxed{2.0}$ Show that $g(x)$ is a contraction in a domain containing one of the two roots of $f(x)$. Determine precisely that domain.

(d) $\boxed{2.0}$ Prove that the sequence $x^{(k+1)} = g(x^{(k)})$ converges to one of the roots of $f(x)$ for any starting value $x^{(0)} \neq 0 \in \mathbb{R}$.

Exercise 2 (3 points)

We want to solve the linear system $Ax = b$ for $x \in \mathbb{R}^2$ by using stationary Richardson iterations:

$$x^{(k)} = x^{(k-1)} + \alpha \left(b - Ax^{(k-1)} \right)$$

using as initial guess the vector $x^{(0)} = [1, 0]^\intercal$, $b = [0, 1]^\intercal$. The matrix A is given by:

$$A = \begin{bmatrix} a & -c \\ -c & a \end{bmatrix} \ , \ a > c > 0.$$

(a) $\boxed{0.5}$ Compute $x^{(1)}$ from $x^{(0)}$ in terms of α, a, c.

(b) $\boxed{1.5}$ Give a value of α in terms of a and/or c so that convergence of the Richardson iterations towards $A^{-1}b$ is ensured. Justify your answers in view of the theory.

(c) $\boxed{1.0}$ The exact solution of the linear system is given by $x^* = [x_1^*, x_2^*]^\intercal$. Find the value of α such that $\|x^{(1)} - x^*\|_2^2$ is minimal. Answer this question by using only the information given and results obtained in this test.